

Unbounded Regime for Circle Maps and Physical Measures for Cherry Flows

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Abstract

We study \mathcal{C}^2 weakly order preserving circle maps with a flat interval. In particular we are interested in the geometry of the mapping near to the singularities at the boundary of the flat interval. Without any assumption on the rotation number we show that the geometry is degenerate when the degree of the singularities is less than or equal to two and become bounded when the degree goes to three. This result allows us to give a description of the physical measures on Cherry flows.

1 Introduction

1.1 Motivation

The principal purpose of this paper is to study the dynamics of a class \mathcal{L} of circle maps of degree one, supposed to be \mathcal{C}^2 everywhere with the exception of two points where they are continuous and such that they are constant on one of the two intervals delimited by these two points. Moreover on a half open neighborhood of these two points the maps can be written as x^ℓ where the real positive number ℓ is called the critical exponent of the function.

The study of this kind of maps has a long history (see [7], [8], [16], [9], [4], [11]). One of the reasons of their investigation is connected to the comprehension of particular flows on the two-dimensional torus, called Cherry flows. In fact the first return map for a Cherry flow is a function belonging to the class \mathcal{L} (for more details see [9], [11], [1]). The first example of such a flow was given by Cherry in 1937 and still now a lot of questions about metric, ergodic and topological properties of Cherry flows remain open.

As the maps we consider are continuous and weakly order preserving, they have a rotation number; this number is the quantity which measures the rate at which an orbit winds around the circle. More precisely, if f is a map in \mathcal{L} and F is a lifting of f to the real line, the rotation number of f is the limit

$$\rho(f) = \lim_{n \rightarrow \infty} \frac{F^n(x)}{n} \pmod{1}.$$

This limit exists for every x and its value is independent to x . Because the dynamics is more interesting, in the discussion that follows and for the rest of this paper we will

assume that the rotation number is irrational. Also, it will often be convenient to identify f with a lift F and subsets of \mathbb{S}^1 with the corresponding subsets of \mathbb{R} .

In this paper we are interested in the study of the geometry of functions in \mathcal{L} near to the boundary points of the flat interval. Without any assumption on the rotation number we discover a change of the geometry depending on the degree of the singularities at the boundary points of the flat interval. In particular, we show that the geometry is bounded when the critical exponent of our maps become greater than 3. This result is particularly interesting because it opens the way for the comprehension of metric and ergodic properties of Cherry flows.

Before we can explain our results more precisely, it is necessary to define our class and fix some notation.

1.2 Assumptions and Notations

Hypotheses.

1. We consider continuous circle endomorphisms f of degree one, at least twice continuously differentiable except for two points (endpoints of the flat interval).
2. The first derivative of f is everywhere positive except for the closure of an open non-degenerate interval U (the flat interval) on which it is equal to zero.
3. Let (a, b) be a preimage of U under the natural projection of the real line on \mathbb{S}^1 . On some right-sided neighborhood of b , f can be represented as

$$h_r \left((x - b)^\ell \right),$$

where h_r is a C^2 -diffeomorphism on an open neighborhood of b . Analogously, there exists a C^2 -diffeomorphism on a left-sided neighborhood of a such that f is of the form

$$h_l \left((a - x)^\ell \right).$$

The real positive number ℓ is called the critical exponent of f .

4. The rotation number $\rho(f) \in (0, 1)$ of f is irrational.

In the future we will assume that $h_r(x) = h_l(x) = x$. It is in fact possible to make a C^2 coordinate changes near a and b that will allow us to replace both h_r and h_l with the identity function.

The class of such a maps will be denoted by \mathcal{L} .

Basic Notations. We will introduce a simplified notation for backward and forward images of the flat interval U . Instead of $f^i(U)$ we will simply write \underline{i} ; for example, $\underline{0} = U$. Thus, for us, underlined positive integers represent points, and underlined non-positive integers represent intervals.

Distance between Points. We denote by $(a, b) = (b, a)$ the shortest open interval between a and b regardless of the order of these two points. The length of that interval in the natural metric on the circle will be denoted by $|a - b|$. Following [4], let us adopt now these conventions of notation:

- $|\underline{-i}|$ stands for the length of the interval $\underline{-i}$.
- Consider a point x and an interval $\underline{-i}$ not containing it. Then the distance from x to the closer endpoint of $\underline{-i}$ will be denoted by $|(x, \underline{-i})|$, and the distance to the more distant endpoint by $|(x, \underline{-i})|$.
- We define the distance between the endpoints of two intervals $\underline{-i}$ and $\underline{-j}$ analogously. For example, $|(\underline{-i}, \underline{-j})|$ denotes the distance between the closest endpoints of these two intervals while $|\underline{[-i, -j]}|$ stands for $|\underline{-i}| + |(\underline{-i}, \underline{-j})|$.

Combinatorics. Let $f \in \mathcal{L}$ and let $\rho(f)$ be the rotation number of f . So, $\rho(f)$ can be written as an infinite continued fraction

$$\rho(f) = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\dots}}},$$

where a_i are positive integers.

If we cut off the portion of the continued fraction beyond the n -th position, and write the resulting fraction in lowest terms as $\frac{p_n}{q_n}$ then the numbers q_n for $n \geq 1$ satisfy the recurrence relation

$$q_{n+1} = a_{n+1}q_n + q_{n-1}; \quad q_0 = 1; \quad q_1 = a_1. \quad (1.1)$$

The number q_n is the number of times we have to iterate the rotation by $\rho(f)$ in order that the orbit of any point makes the closest return so far to the point itself (see Chapter I, Sect. I in [3]).

1.3 Discussion and Statement of the Results

As stressed before, in this paper we are interested in the study of the geometry of functions in \mathcal{L} near to the boundary points of the flat interval. This quantity is measured by the sequence of scalings

$$\tau_n := \frac{|(\underline{0}, q_n)|}{|(\underline{0}, q_{n-2})|}.$$

When $\tau_n \rightarrow 0$ we say that the geometry of the mapping is ‘degenerate’. When τ_n is bounded away from zero we say that the geometry is ‘bounded’.

The same problem was analyzed in [4] for functions in \mathcal{L} with rotation number of bounded type ($\sup_i a_i < \infty$) and with negative Schwarzian derivative¹. This last assumption was then removed in [11]. The authors prove that the geometry is degenerate when the critical exponent is less than or equal to 2 and become bounded when the critical exponent passes 2. So, a phase transition occurs in the dynamics of the system depending on the degree of the singularities at the boundary points of the flat interval.

This result suggests us the natural problem of investigating the unbounded regime. In this case it becomes more delicate to make conjectures; surprises often occur due to the presence of underlying parabolic phenomena. The main result we have obtained is the following:

Theorem 1.2. *Let f be a function of the class \mathcal{L} with critical exponent $\ell > 1$:*

¹The Schwarzian derivative of a function f is defined to be $Sf(x) := \frac{f'''(x)}{f'(x)} - \frac{3}{2} \left(\frac{f''(x)}{f'(x)} \right)^2$.

1. If $\ell \leq 2$, then the sequence $(\tau_n)_{n \in \mathbb{N}}$ tends to zero at least exponentially fast.
2. If $\ell \geq 3$, the sequence $(\tau_n)_{n \in \mathbb{N}}$ is bounded away from zero.

Without any assumption on the rotation number we prove that the geometry near to the boundary points of the flat interval is degenerate when the critical exponent is less than or equal to 2 and become bounded when the critical exponent goes to 3. It rests unknown what happens between 2 and 3.

The difficulty of the problem comes from the presence of parabolic phenomena that generate accumulation of constants which is not always easy to control. In fact the main idea of the proof is to find a recursive formula for the sequence τ_n and to study its convergence. The accumulation of constants appear basically everywhere: both in the recurrence as well as in the study of the convergence. This fact suspect us that the problem could be real, not only technical.

Moreover this result remains the first one to be valid for functions with any rotation number and has many interesting and strong applications. One of these concerns Cherry flows, interpreted as flows on the torus without closed trajectories and with two singularities, a saddle point and a repulsive point, both hyperbolic. Theorem 1.2 allows us to give a description of physical measures on Cherry flows. This kind of measures have basin of attraction of positive Lebesgue measure. For this reason they are particularly interesting for physicists.

This problem was firstly studied in [13] where the authors show that the existence of ergodic invariant probability measures for Cherry flows depends on the divergence² at the saddle point. They prove that when the divergence is negative the Dirac deltas at the singularities are the only ergodic invariant probability measures and the Dirac delta at the saddle point is the physical one for the flow. They show also that when the divergence becomes positive a third measure ν , supported on the quasi-minimal set, appears and they conjecture that this is the physical measure for the flow.

We solve this conjecture and we prove that if the divergence at the saddle point is positive and the Cherry flow has rotation number³ of bounded type:

Theorem 1.3. *Let ϕ be a Cherry flow with eigenvalues at the saddle point $\lambda_1 > 0 > \lambda_2$. If $\lambda_1 > -\lambda_2$ and ϕ has rotation number of bounded type, the measure ν defined above is the physical measure for ϕ with attraction basin having full Lebesgue measure.*

If the divergence at the saddle point become strictly positive, without any assumption on the rotation number, we have:

Theorem 1.4. *Let ϕ be a Cherry flow with eigenvalues $\lambda_1 > 0 > \lambda_2$. If $\lambda_1 \geq -3\lambda_2$, the ergodic invariant probability measure ν supported on the quasi-minimal set is the physical measure for ϕ with attractive basin having full Lebesgue measure.*

Finally, in the bounded regime, we have a complete picture of physical measures on Cherry flows. It remains to understand what happens in the unbounded regime when the ratio of the eigenvalues of the saddle point is between 1 and 3. Reading the proof of Theorem 1.4 the reader can verify that this problem is strictly connected with the

²Observe that the divergence of a Cherry flow ϕ at the saddle point p_s is exactly the sum of the eigenvalues of the flow ϕ at p_s .

³Cherry flow has a well defined rotation number $\rho \in [0, 1)$ equal to the rotation number of its first return map to any global Poincaré section (it is easy to see that ρ does not depend on the choice of a Poincaré section).

accumulation of constants explained before which doesn't allow us to have an optimal recursive formula for the sequence τ_n .

In order to give a complete panorama of the applications of Theorem 1.2, we want to underline that, following the strategies in [11], we are now able to give a characterization of the Hausdorff dimension of the quasi-minimal set for Cherry flows in the general case of unbounded regime. This calculation is still work in progress.

2 Technical Tools

2.1 Distortion Techniques

The main ingredient in the proof of the principal result of this paper is the control of the distortion of iterates of maps in \mathcal{L} . We will use two different cross-ratios, **Cr** and **Poin**.

Definition 2.1. *If $a < b < c < d$ are four points on the circle, then we can define their cross-ratio **Cr** by:*

$$\mathbf{Cr}(a, b, c, d) := \frac{|b - a||d - c|}{|c - a||d - b|},$$

*and their cross-ratio **Poin** by:*

$$\mathbf{Poin}(a, b, c, d) := \frac{|d - a||b - c|}{|c - a||d - b|}.$$

Now we analyze the distortion of this two kinds of cross-ratios.

Diffeomorphisms with negative Schwarzian derivative increase cross-ratio **Poin**:

$$\mathbf{Poin}(f(a), f(b), f(c), f(d)) > \mathbf{Poin}(a, b, c, d).$$

In general, without the assumption of negative Schwarzian, the following holds:

Theorem 2.2. *Let f be a C^2 map with no flat critical points. There exists a bounded increasing function $\sigma : [0, \infty) \rightarrow \mathbb{R}_+$ with $\sigma(t) \rightarrow 0$ as $t \rightarrow 0$ with the following property. Let $[b, c] \subset [a, d]$ be intervals such that $f^n|_{[a, d]}$ is a diffeomorphism. Then*

$$\mathbf{Poin}(f^n(a), f^n(b), f^n(c), f^n(d)) \geq \exp\{-\sigma(\tau) \sum_{i=0}^{n-1} |f^i([a, b])|\} \mathbf{Poin}(a, b, c, d),$$

where $\tau = \max_{i=0, \dots, n-1} |f^i([c, d])|$.

The proof of Theorem 2.2 can be found in [15].

Here, we formulate the result which enables us to control a growth of the iterates of cross-ratios **Cr** even if maps are no longer homeomorphisms with negative Schwarzian nor are invertible.

The reader can refer to [14] for the general case and to [5] for our situation.

Consider a chain of quadruples

$$\bigcup_{i=0}^n \{(a_i, b_i, c_i, d_i)\}$$

such that each is mapped onto the next by the map f . If the following conditions hold:

- There exists an integer $k \in \mathbb{N}$, such that each point of the circle belongs to at most k of the intervals (a_i, d_i) .
- Intervals (b_i, c_i) do not intersect $\underline{0}$.

Then, there exists a constant $K > 0$, independent on the set of quadruples, such that:

$$\log \frac{\mathbf{Cr}(a_n, b_n, c_n, d_n)}{\mathbf{Cr}(a_0, b_0, c_0, d_0)} \leq K$$

In order to control the distortion of the iterates of our maps we will also frequently use the following proposition which is a corollary of Koebe principle in [6].

Proposition 1. *Let f be a function in \mathcal{L} and let $J \subset T$ be two intervals of the circle. Suppose that, for some $n \in \mathbb{N}$*

- f^n is a diffeomorphism on T ,
- $\sum_{i=0}^{n-1} |f^i(J)|$ is bounded,
- $|f^n(J)| \leq K \text{dist}(f^n(J), \partial f^n(T))$ with K a positive constant.

Then, there exists a constant C such that, for every two intervals A and B in J

$$\frac{|f^n(A)|}{|f^n(B)|} \geq C \frac{|A|}{|B|}.$$

2.2 Continued Fractions and Partitions

Let $f \in \mathcal{L}$. Since f is order-preserving and has no periodic points, there exists an order-preserving and continuous map $h : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ such that $h \circ f = R_\rho \circ h$, where ρ is the rotation number of f and R_ρ is the rotation by ρ . In particular, the order of points in an orbit of f is the same as the order of points in an orbit of R_ρ . Therefore, results about R_ρ can be translated into results about f , via the semiconjugacy h .

We can build the so called dynamical partitions \mathcal{P}_n of \mathbb{S}^1 to study the geometric properties of f , see [5]. The partition \mathcal{P}_n is generated by the first $q_n + q_{n+1}$ preimages of U and consists of

$$\mathcal{I}_n := \{ \underline{-i} : 0 \leq i \leq q_{n+1} + q_n - 1 \},$$

together with the gaps between these intervals.

There are two kinds of gaps:

- The ‘long’ gaps are of the form

$$I_i^n := f^{-i}(I_0^n), i = 0, 1, \dots, q_{n+1} - 1$$

where I_0^n is the interval between $\underline{-q_n}$ and $\underline{0}$ for n even or the interval between $\underline{0}$ and $\underline{-q_n}$ for n odd.

- The ‘short’ gaps are of the form

$$I_i^{n+1} := f^{-i}(I_0^{n+1}), i = 0, 1, \dots, q_n - 1$$

where I_0^{n+1} is the interval between $\underline{0}$ and $\underline{-q_{n+1}}$ for n even or the interval between $\underline{-q_{n+1}}$ and $\underline{0}$ for n odd.

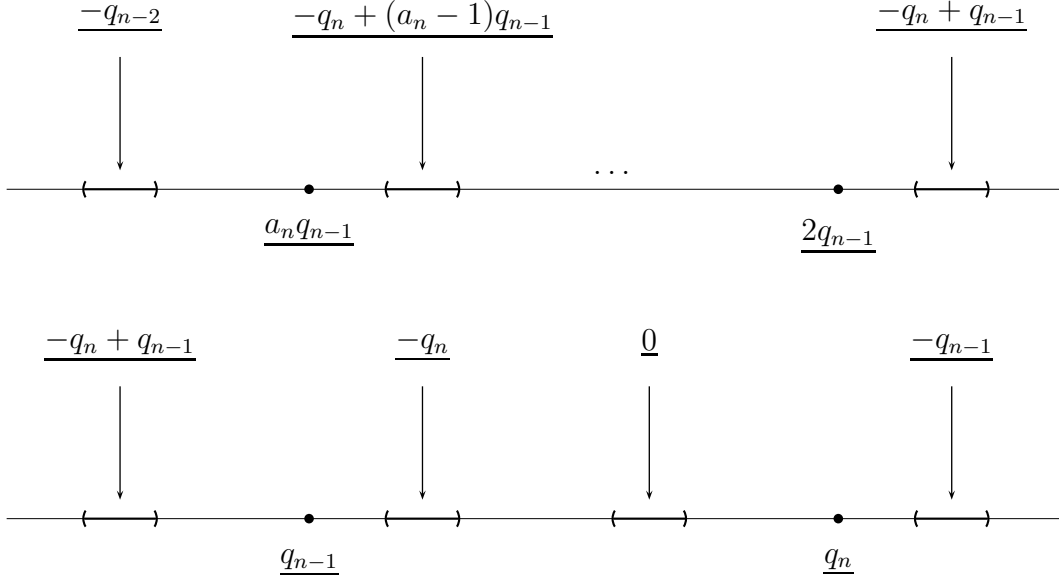


Figure 1: Structure of the dynamical partition \mathcal{P}_{n-1} for n even and $a_n > 1$.

We will briefly explain the structure of the partitions. Take two consecutive dynamical partitions of order n and $n+1$. The latter is clearly a refinement of the former. All ‘short’ gaps of \mathcal{P}_n become ‘long’ gaps of \mathcal{P}_{n+1} while all ‘long’ gaps of \mathcal{P}_n split into a_{n+2} preimages of U and a_{n+2} ‘long’ gaps and one ‘short’ gap of the next partition \mathcal{P}_{n+1} :

$$I_i^n = \bigcup_{j=1}^{a_{n+2}} f^{-i-q_n-jq_{n+1}}(U) \cup \bigcup_{j=0}^{a_{n+2}-1} I_{i+q_n+jq_{n+1}}^{n+1} \cup I_i^{n+2}. \quad (2.3)$$

Several of the proofs in the following will depend strongly on the relative positions of the points and intervals of \mathcal{P}_n . In reading the proofs the reader is advised to keep the Figure 1 in mind, which show some of these objects near the flat interval $\underline{0}$.

We state a standard fact and few results from [4] which will be used frequently in the paper.

Fact 2.4. *Let $f \in \mathcal{L}$ and let x, y, z be three points of the circle with y between x and z such that, of the three, the point x is closest to the flat interval. If f is a diffeomorphism on (x, z) , the following inequality holds:*

$$\frac{|f(z) - f(y)|}{|f(z) - f(x)|} \leq K \frac{|z - y|}{|z - x|},$$

where K is a positive uniform constant.

Proposition 2. *There exists a constant $C > 0$, such that for all $n \in \mathbb{N}$ and $m \in \mathbb{N}$, if $J = f^{-m}(U)$ is a preimage of the flat interval U which belongs to the dynamical partition \mathcal{P}_n and I is one of the two gaps adjacent to J , then:*

$$\frac{|J|}{|I|} \geq C$$

Corollary 2.5. *The lengths of gaps of the dynamical partition \mathcal{P}_n tend to zero at least exponentially fast when $n \rightarrow \infty$.*

The proofs of Proposition 2 and Corollary 2.5 can be found in [4], pag. 606-607.

Standing assumption. In the following we will always work with functions in \mathcal{L} which have critical exponent $\ell > 1$ and irrational rotation number.

3 Proof of the First Part of Theorem 1.2

In this section we will always work with the following sequence:

$$\alpha_n = \frac{|(-q_n, \underline{0})|}{|[-q_n, \underline{0})|}.$$

Since $\forall n \in \mathbb{N}$, $\alpha_n > \tau_n$, we shall prove the first part of Theorem 1.2 for the sequence $(\alpha_n)_{n \in \mathbb{N}}$.

Let

$$\sigma_n = \frac{|(\underline{0}, q_n)|}{|(\underline{0}, q_{n-1})|}$$

and

$$s_n = \frac{|[-q_{n-2}, \underline{0})|}{|\underline{0}|}.$$

We have the following Theorem:

Theorem 3.1. *There exists a natural number $N \in \mathbb{N}$, such that for $n > N$ we have the following inequality:*

$$(\alpha_n)^l \leq \prod_{i=0}^{a_n-1} K_{i,n} C_n \tilde{M}_n(l) \alpha_{n-2}^2 \quad (3.2)$$

where for all $i \in \{0, \dots, a_n - 1\}$, if we denote:

$$\tau_{i,n} = \max_{j \in \{0, \dots, q_{n-1}-2\}} \left| f^j((i q_{n-1} + 1, -q_{n-1} + 1]) \right|$$

and

$$\rho_n = \max_{j \in \{0, \dots, q_{n-2}-2\}} \left| f^j((a_n q_{n-1} + 1, \underline{1}) \right|,$$

then

$$\begin{aligned} K_{i,n} &= e^{\sigma(\tau_{i,n}) \sum_{j=0}^{q_{n-1}-2} |f^j(-q_n + i q_{n-1} + 1)|}, \\ C_n &= e^{\sigma(\rho_n) \sum_{j=0}^{q_{n-2}-2} |f^j(-q_{n-2} + 1)|} \end{aligned} \quad (3.3)$$

and

$$\tilde{M}_n(l) = s_{n-1}^2 \cdot \frac{2}{l} \cdot \left(\frac{1}{1 + \sqrt{1 - \frac{2(l-1)}{l} C_n s_{n-1} \alpha_{n-1}}} \right) \cdot \frac{1}{1 - \alpha_{n-2}} \cdot \frac{\sigma_n}{\sigma_{n-2}}.$$

Proof. The proof is exactly the same of Theorem A.2 in [11] (pag. 150). In fact, the author doesn't use any assumption on the rotation number. \square

We prove now the convergence of the sequence $(\alpha_n)_{n \in \mathbb{N}}$.

In [11] (see Lemma A.12 and continuation, pag. 153-154), without any assumption on the rotation number, the author prove that $\prod_{m=0}^n C_m$ converges and that $\prod_{m=0}^n \tilde{M}_m$ tends to zero. It rest to study the convergence of the product

$$\prod_{m=0}^n \prod_{i=0}^{a_m-1} K_{i,m}$$

which is assured by the following Lemma:

Lemma 3.4. *There exists $0 < \lambda < 1$ such that, for all $i \in \{0, \dots, a_n - 1\}$ and for m big enough,*

$$\prod_{i=0}^{a_m-1} K_{i,m} \leq e^{\sigma(\lambda^{m-3})\lambda^{m-2}}.$$

Proof. By the order of the preimages of the flat interval on the circle (see Subsection 2.2) and the definition of $K_{i,m}$ (see 3.3) we can observe that:

1. each interval $f^j(\underline{(iq_{m-1} + 1, -q_{m-1} + 1]})$ is contained in a gap of the partition \mathcal{P}_{m-3} ,
2. for all i , $\sum_{j=0}^{q_{m-1}-2} \left| f^j(\underline{-q_m + iq_{m-1} + 1}) \right|$ is contained in a gap of \mathcal{P}_{m-1} and each two sums $\sum_{j=0}^{q_{m-1}-2} \left| f^j(\underline{-q_m + iq_{m-1} + 1}) \right|$ and $\sum_{j=0}^{q_{m-1}-2} \left| f^j(\underline{-q_m + i'q_{m-1} + 1}) \right|$ are disjointed. More the total sum $\sum_{i=0}^{a_m-1} \sum_{j=0}^{q_{m-1}-2} \left| f^j(\underline{-q_m + iq_{m-1} + 1}) \right|$ is contained in a gap of the partition \mathcal{P}_{m-2} .

Since the lengths of the gaps of the partition \mathcal{P}_m tend to zero at least exponentially fast by Corollary 2.5, then there exists $0 < \lambda < 1$ such that, for m big enough,

$$\sigma \left(\max_{j \in \{0, \dots, q_{m-1}-2\}} |f^j(\underline{(iq_{m-1} + 1, -q_{m-1} + 1]})| \right) < \sigma(\lambda^{m-3})$$

and

$$\sum_{i=0}^{a_m-1} \sum_{j=0}^{q_{m-1}-2} \left| f^j(\underline{-q_m + iq_{m-1} + 1}) \right| < \lambda^{m-2}.$$

Finally $\prod_{i=0}^{a_m-1} K_{i,m}$ is equal to

$$\exp \left(\sum_{i=0}^{a_m-1} \sigma \left(\max_{j \in \{0, \dots, q_{m-1}-2\}} |f^j(\underline{(iq_{m-1} + 1, -q_{m-1} + 1]})| \right) \sum_{j=0}^{q_{m-1}-2} \left| f^j(\underline{-q_m + iq_{m-1} + 1}) \right| \right)$$

which is strictly less than

$$\exp \left(\sigma(\lambda^{m-3})\lambda^{m-2} \right).$$

□

In conclusion, for $l \leq 2$ the inequality (3.2) implies that the sequence $(\alpha_n)_{n \in \mathbb{N}}$ tends to zero at least exponentially fast. Since $\alpha_n > \tau_n$, we have the same result for the sequence $(\tau_n)_{n \in \mathbb{N}}$.

4 Proof of the Second Part of Theorem 1.2

4.1 Some Technical Lemmas

We present some technical lemmas which we need for the proof of the main theorem.

Standing assumption. Because of the symmetry of the functions in \mathcal{L} we assume always that $n \in \mathbb{N}$ is even. The case $n \in \mathbb{N}$ odd is completely analogue.

Lemma 4.1. *There exists a constant $K > 0$, such that the fraction*

$$\frac{|(2q_{n+1}, \underline{q_{n+1}})|}{|(\underline{2q_{n+1}}, \underline{0})|} > K > 0.$$

Proof. See Lemma 1.2 in [4]. □

Lemma 4.2. *There exists a constant $K > 0$, such that for n large enough the fraction*

$$\frac{|-q_n - q_{n+1}|}{|(\underline{-q_n - q_{n+1}}, \underline{0})|} > K > 0.$$

Proof. The reader can keep in mind Figure 2.

By Fact 2.4 there exists a constant $K_1 > 0$ such that

$$\begin{aligned} \frac{|-q_n - q_{n+1}|}{|(\underline{-q_n - q_{n+1}}, \underline{0})|} &\geq K_1 \frac{|-q_n - q_{n+1} + 1|}{|(\underline{-q_n - q_{n+1} + 1}, \underline{1})|} \geq \\ &\geq K_1 \frac{|-q_n - q_{n+1} + 1|}{|(\underline{-q_n - q_{n+1} + 1}, \underline{-q_{n+1} + 1})|} \frac{|-q_{n+1} + 1|}{|(\underline{-q_n - q_{n+1} + 1}, \underline{-q_{n+1} + 1})|}. \end{aligned}$$

We apply $f^{q_{n+1}-1}$. By the properties of distortion of cross-ratio **Cr**, there exists a positive constant K_2 such that:

$$\frac{|-q_n - q_{n+1}|}{|(\underline{-q_n - q_{n+1}}, \underline{0})|} \geq K_1 K_2 \frac{|-q_n|}{|(\underline{-q_n}, \underline{0})|} \frac{|0|}{|(\underline{-q_n}, \underline{0})|} \geq K_1 K_2 \frac{|-q_n|}{|(\underline{-q_n}, \underline{0})|}. \quad (4.3)$$

Using Proposition 2 the proof is complete. □

Lemma 4.4. *There exists a constant $K > 0$ such that, for n large enough,*

$$\frac{|-q_{n-1} - q_n + 1|}{|(\underline{-q_n + 1}, \underline{1})|} \geq K.$$

Proof. The reader can keep in mind Figure 3.

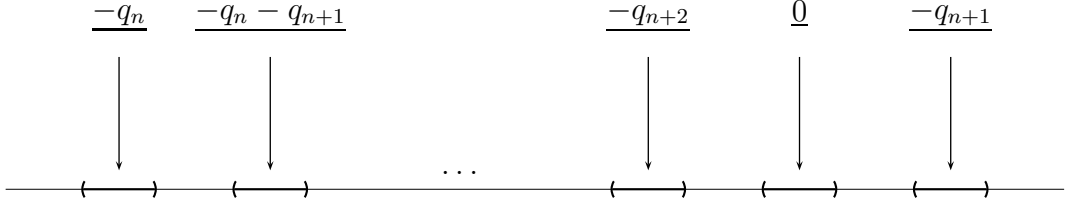


Figure 2:

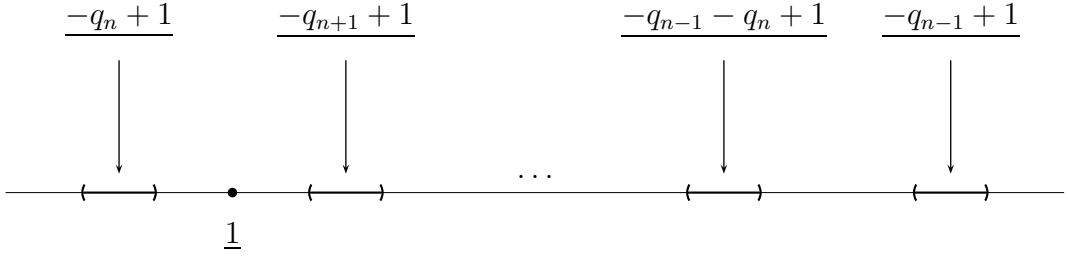


Figure 3:

We have:

$$\begin{aligned}
 \frac{|\underline{-q_{n-1} - q_n + 1}|}{|(\underline{-q_n + 1}, \underline{1})|} &\geq \frac{|\underline{-q_{n-1} - q_n + 1}|}{|(\underline{-q_n + 1}, \underline{-q_{n-1} - q_n + 1})|} \\
 &\geq \frac{|\underline{-q_n + 1}|}{|[\underline{-q_n + 1}, \underline{-q_{n-1} - q_n + 1})|} \frac{|\underline{-q_{n-1} - q_n + 1}|}{|(\underline{-q_n + 1}, \underline{-q_{n-1} - q_n + 1})|}.
 \end{aligned} \tag{4.5}$$

After $(q_n - 1)$ iterates, by the distortion properties of cross-ratio \mathbf{Cr} , there exists a constant $K_1 > 0$ such that:

$$\frac{|\underline{-q_{n-1} - q_n + 1}|}{|(\underline{-q_n + 1}, \underline{1})|} \geq K_1 \frac{|\underline{0}|}{|[\underline{0}, \underline{-q_{n-1}})|} \frac{|\underline{-q_{n-1}}|}{|(\underline{0}, \underline{-q_{n-1}})|}$$

which, for n large enough, is minored by a positive constant (Proposition 2). \square

For all n and for all $i \in \{0, \dots, a_{n+2} - 1\}$ we define (see Figure 4) :

$$\beta_n(i) = \frac{|(\underline{-q_n - (a_{n+2} - i)q_{n+1}}, \underline{0})|}{|[\underline{-q_n - (a_{n+2} - i)q_{n+1}}, \underline{0})|}$$

and

$$\gamma_n(i) = \frac{|[\underline{-q_n - (a_{n+2} - i)q_{n+1}}, \underline{0})|}{|(\underline{-q_n - (a_{n+2} - (i + 1))q_{n+1}}, \underline{0})|}$$

and we prove the following lemma:

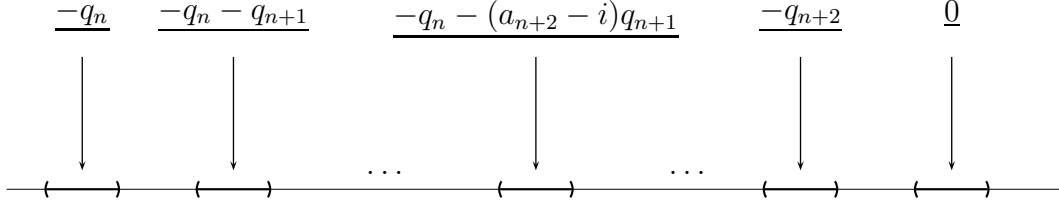


Figure 4:

Lemma 4.6. *There exists a constant $K > 0$, such that for all $i \in \{1, \dots, a_{n+2} - 2\}$, we have:*

$$(\beta_n(i))^\ell \geq K\beta_n(i+1).$$

We observe that this lemma makes sense under the assumption that $a_{n+2} \notin \{1, 2\}$.

Proof. We apply f to the intervals defining $\beta_n(i)$ and we obtain, for large n ,

$$\beta_n(i)^\ell = \frac{|(-q_n - (a_{n+2} - i)q_{n+1} + 1, \underline{1})|}{|[-q_n - (a_{n+2} - i)q_{n+1} + 1, \underline{1})|}.$$

For all $i \in \{1, \dots, a_{n+2} - 2\}$ we apply Proposition 1 to

- $T = [-q_n - q_{n+1} + 1, -q_{n+1} + 1]$,
- $J = (-q_n - q_{n+1} + 1, -q_{n+1} + 1)$,
- $f^{q_{n+1}-1}$.

We notice that the hypotheses are satisfied:

- $f^{q_{n+1}-1}$ is a diffeomorphism on T ,
- the intervals $f^j(J)$ for $j \in \{1, \dots, q_{n+1} - 2\}$ are disjoint,
- by Proposition 2, for n large enough, there exists a positive constant K_1 such that

$$|f^{q_{n+1}-1}(J)| = |(-q_n, \underline{0})| \leq K_1 |(-q_n)| = K_1 \text{dist}(f^{q_{n+1}-1}(J), \partial f^{q_{n+1}-1}(T)).$$

Then we find a uniform constant $K_2 > 0$ such that:

$$\beta_n(i)^\ell = \frac{|(-q_n - (a_{n+2} - i)q_{n+1} + 1, \underline{1})|}{|[-q_n - (a_{n+2} - i)q_{n+1} + 1, \underline{1})|} \geq K_2 \frac{|(-q_n - (a_{n+2} - (i+1))q_{n+1}, \underline{q_{n+1}})|}{|[-q_n - (a_{n+2} - (i+1))q_{n+1}, \underline{q_{n+1}})|}. \quad (4.7)$$

Since the numerator of (4.7) contains the interval $(2q_{n+1}, \underline{q_{n+1}})$, for Lemma 4.1 we can conclude that $(\beta_n(i))^\ell$ is greater than a positive constant multiplied by $\beta_n(i+1)$. \square

Lemma 4.8. *There exist two constant $K_1 > 0$ and $K_2 > 0$ such that:*

1. for all $0 \leq i \leq a_{n+2} - 2$, $(\gamma_n(i))^\ell \geq K_1\gamma_n(i+1)$,

$$2. \gamma_n(a_{n+2} - 1) \geq K_2.$$

Proof. In order to prove the point (2) it is sufficient to observe that:

$$\gamma_n(a_{n+2} - 1) = \frac{|[-q_n - q_{n+1}, \underline{0})|}{|(\underline{-q_n}, \underline{0})|}$$

which is greater than a positive uniform constant (see Proposition 2).

In order to prove point (1) we before apply f to intervals defining $\gamma_n(i)$ and after Proposition 1 to

- $T = [-q_n - q_{n+1} + 1, -q_{n+1} + 1]$,
- $J = (\underline{-q_n - q_{n+1} + 1}, \underline{-q_{n+1} + 1})$,
- $f^{q_{n+1}-1}$.

Like in Lemma 4.6, the hypotheses are satisfied, then there exists a constant $K_1 > 0$ such that, for all i , $(\gamma_n(i))^\ell \geq K_1 \gamma_n(i + 1)$. \square

In the following, in order to simplify notation, we note $\beta_n = \beta_n(a_{n+2} - 1)$ and $\gamma_n = \gamma_n(a_{n+2} - 1)$.

4.2 The Central Part of the Proof

We recall that

$$\tau_n = \frac{|(\underline{0}, \underline{q_n})|}{|(\underline{0}, \underline{q_{n-2}})|},$$

$$\alpha_n = \frac{|(\underline{-q_n}, \underline{0})|}{|(\underline{-q_n}, \underline{0})|}$$

and we introduce a new parameter which measures the relative size of α_n and τ_n ,

$$k_n = \frac{|(\underline{0}, \underline{q_n})|}{|(\underline{0}, \underline{-q_{n-1}})|}.$$

Remark 4.9. We recall that the point $\underline{q_{n-2}}$ is situated in the gap between $\underline{-q_{n-1}}$ and $\underline{-q_{n-1} + q_{n-2}}$ of the dynamical partition \mathcal{P}_{n-2} .

Then, by Proposition 2, there exists a constant $K > 0$ such that

$$k_n \geq \frac{\tau_n}{\alpha_{n-1}} \geq K k_n.$$

Finally, $\frac{\tau_n}{\alpha_{n-1}}$ and k_n are comparable.

To complete the proof of the second part of Theorem 1.2 is necessary to find a bound for the sequence $(\alpha_n)_{n \in \mathbb{N}}$ and $(k_n)_{n \in \mathbb{N}}$. For this reason we prove the following propositions.

Proposition 3. *There exists a positive constant K such that, for n large enough*

$$k_n \geq K \beta_{n-1}^{\left(\frac{1-\ell^{-a_{n+1}-1}}{1-\ell^{-1}}\right)}.$$

Proof. By Proposition 2 there exists a uniform constant $K_1 > 0$ such that

$$k_n \geq K_1 \frac{|(\underline{0}, \underline{-q_{n-1} - (a_{n+1} - 1)q_n})|}{|(\underline{0}, \underline{-q_{n-1}})|}. \quad (4.10)$$

For all $i \in \{1, \dots, a_{n+1} - 1\}$ and for all $j \in \{2, \dots, a_{n+1} - 2\}$, we multiply and divide alternatively by $|(\underline{0}, \underline{-q_{n-1} - (a_{n+1} - i)q_n})|$ and by $|(\underline{0}, \underline{-q_{n-1} - (a_{n+1} - j)q_n})|$ to obtain that:

$$k_n \geq K_1 \beta_{n-1}(1) \gamma_{n-1}(2) \beta_{n-1}(2) \dots \gamma_{n-1}(a_{n+1} - 2) \beta_{n-1}(a_{n+1} - 1) \frac{|(\underline{0}, \underline{-q_{n-1} - q_n})|}{|(\underline{0}, \underline{-q_{n-1}})|}.$$

By Proposition 2, there exists a positive constant K_2 such that $\frac{|(\underline{0}, \underline{-q_{n-1} - q_n})|}{|(\underline{0}, \underline{-q_{n-1}})|} > K_2$. We apply Lemma 4.6, Lemma 4.8 and we have:

$$k_n \geq K_3 \beta_{n-1} \beta_{n-1}^{\frac{1}{\ell}} \dots \beta_{n-1}^{\left(\frac{1}{\ell^{a_{n+1}-2}}\right)} = K_3 \beta_{n-1}^{\left(\frac{1-\ell^{-a_{n+1}+1}}{1-\ell^{-1}}\right)}. \quad (4.11)$$

where K_3 is a positive constant.

It remains to study the case $a_{n+1} = 1$ and $a_{n+1} = 2$ for which we can't use Lemma 4.6.

We assume that $a_{n+1} = 1$. In this case the gap in the right of $\underline{-q_{n+1}}$ is $(\underline{-q_{n+1}}, \underline{-q_{n-1}})$ and then by Proposition 2 there exists $K_4 > 0$ such that $k_n \geq K_4$.

We find the same inequality that (4.11) in the specific case $a_{n+1} = 1$.

If $a_{n+1} = 2$ we proceed like in the general case up to obtain (like in (4.10)) that:

$$k_n \geq K_1 \frac{|(\underline{0}, \underline{-q_{n-1} - q_n})|}{|(\underline{0}, \underline{-q_{n-1}})|}$$

which by Proposition 2 is greater than a uniform positive constant multiplied by β_{n-1} .

The proof of the proposition is then complete. \square

Proposition 4. *There exists a constant $K > 0$ such that, for n large enough*

$$\alpha_n \geq K \beta_{n-1}^{\left(\frac{1-\ell^{-a_{n+1}}}{1-\ell^{-1}}\right)} \beta_{n-2}^{\ell^{-a_{n+1}}}.$$

Proof. If n is large enough, $(\alpha_n)^\ell$ is equal to the fraction

$$\frac{|(\underline{-q_n + 1}, \underline{1})|}{|(\underline{-q_n + 1}, \underline{1})|}$$

which is greater than the product of followings three fractions:

$$\begin{aligned} \xi_1 &= \frac{|(\underline{-q_n + 1}, \underline{1})|}{|(\underline{-q_n + 1}, \underline{-q_{n-1} - q_n + 1})|}, \\ \xi_2 &= \frac{|(\underline{-q_n + 1}, \underline{-q_{n-1} - q_n + 1})|}{|(\underline{-q_n + 1}, \underline{-q_{n-1} + 1})|}, \\ \xi_3 &= \frac{|(\underline{-q_n + 1}, \underline{-q_{n-1} + 1})|}{|(\underline{-q_n + 1}, \underline{-q_{n-1} + 1})|}. \end{aligned}$$

We focus on each fraction separately.

1st step. We prove that $\xi_1 \geq K_1 \beta_{n-1}^{\left(\frac{1-\ell^{-a_n+1}-1}{1-\ell^{-1}}\right)}$.

We apply Proposition 1 to

- $T = [\underline{-q_n + 1}, \underline{-q_n - q_{n-1} + 1}]$,
- $J = (\underline{-q_n + 1}, \underline{-q_n - q_{n-1} + 1})$,
- f^{q_n-1} .

Like in the previous lemmas the hypothesis are satisfied, then there exists a positive constant C_1 such that:

$$\xi_1 \geq C_1 \frac{|(\underline{0}, \underline{q_n})|}{|(\underline{0}, \underline{-q_{n-1}})|}. \quad (4.12)$$

We apply now Proposition 3 and we find the wanted estimate.

2nd step. We prove that $\xi_2 \geq K_2 \beta_{n-1}^\ell$

By Proposition 2 and Lemma 4.4 we have two positives constants C_2 and C_3 such that

$$\xi_2 \geq C_2 \frac{|(\underline{1}, \underline{-q_{n-1} - q_n + 1})|}{|(\underline{-q_n + 1}, \underline{-q_{n-1} - q_n + 1})|} \geq C_2 C_3 \frac{|(\underline{1}, \underline{-q_{n-1} - q_n + 1})|}{|(\underline{1}, \underline{-q_{n-1} - q_n + 1})|} \quad (4.13)$$

and this last fraction is exactly β_{n-1}^ℓ .

3rd step. We prove that $\xi_3 \geq K_3 \beta_{n-2}^{\ell^{-a_n+2}}$.

We apply Lemma 4.6 and Proposition 1 to

- $T = [\underline{-q_{n-2} - q_{n-1} + 1}, \underline{-q_{n-1} + 1}]$,
- $J = (\underline{-q_{n-2} - q_{n-1} + 1}, \underline{-q_{n-1} + 1})$,
- $f^{q_{n-1}-1}$.

and we find, under the assumption that $a_n \notin \{1, 2\}$, two positives constants $C_4 > 0$ and $C_5 > 0$ such that:

$$\xi_3 \geq C_4 \frac{|(\underline{-q_n + q_{n-1}}, \underline{0})|}{|(\underline{-q_n + q_{n-1}}, \underline{0})|} \geq C_4 \beta_{n-2}(1) \geq C_4 C_5 \beta_{n-2}^{\ell^{-a_n+2}}. \quad (4.14)$$

If $a_n = 1$, we obtain exactly the same estimate, in fact, in this case $\underline{-q_n + 1} = \underline{-q_{n-2} - q_{n-1} + 1}$ and by Lemma 4.4 there exists a constant $C_6 > 0$ such that

$$\xi_3 \geq C_6 \beta_{n-2}^\ell.$$

If $a_n = 2$, we proceed like the general case until the first inequality (4.14). Now it is sufficient to observe that, in this case,

$$\frac{|(-q_n + q_{n-1}, \underline{0})|}{|[-q_n + q_{n-1}, \underline{0})|} = \beta_{n-2}.$$

Finally, using the estimates obtained for ξ_1 , ξ_2 and ξ_3 we have

$$\alpha_n^\ell \geq K \beta_{n-1}^{\left(\frac{1-\ell^{-a_n+1}+1}{1-\ell^{-1}}\right)} \beta_{n-1}^\ell \beta_{n-2}^{\ell^{-a_n+2}}$$

therefore

$$\alpha_n \geq K^{\frac{1}{\ell}} \beta_{n-1}^{\ell \left(\frac{1-\ell^{-a_n+1}}{\ell^{-1}}\right)} \beta_{n-2}^{\ell^{-a_n+1}}.$$

□

By Remark 4.9, Proposition 3 and Proposition 4, in order to find a lower bound for τ_n , it is necessary to study the sequence β_n . Hence the following propositions:

Proposition 5. *There exists a positive constant K such that, for n large enough*

$$\beta_n \geq K \beta_{n-1}^{\frac{1}{\ell}} \alpha_n^{\frac{1}{\ell}}$$

Proof. We start to apply Proposition 1 to

- $T = [-q_n + 1, -q_{n-1} - q_n + 1]$,
- $J = (-q_n + 1, -q_{n-1} - q_n + 1)$,
- f^{q_n-1} .

Then, for n large enough, there exists a constant $C_1 > 0$ such that

$$\beta_n^\ell = \frac{|(-q_n - q_{n+1} + 1, \underline{1})|}{|[-q_n - q_{n+1} + 1, \underline{1})|} \geq C_1 \frac{|(-q_{n+1}, q_n)|}{|[-q_{n+1}, q_n)|}.$$

Multiplying and diving by $|(-q_{n+1}, -q_{n+1} + q_n)|$, we find that β_n^ℓ is greater than (up to a constant) the product of the following two quantities:

$$\eta_1 = \frac{|(-q_{n+1}, q_n)|}{|(-q_{n+1}, -q_{n+1} + q_n)|}, \eta_2 = \frac{|(-q_{n+1}, -q_{n+1} + q_n)|}{|[-q_{n+1}, q_n)|}.$$

We focus our attention on this two quantities.

1st step. We prove that $\eta_1 \geq K_1$.

We use Proposition 1 to

- $T = [-q_{n+1}, -q_{n+1} + q_n]$,
- $J = (-q_{n+1}, -q_{n+1} + q_n)$,

$$- f^{q_{n+1}-q_n}$$

and we find a constant $C_2 > 0$ such that

$$\eta_1 \geq C_2 \frac{|(-q_n, q_{n+1})|}{|(-q_n, 0)|} \geq C_2 \frac{|(-q_n, -q_n - q_{n+1})|}{|(-q_n, 0)|} \geq C_2 C_3 \frac{|-q_n - q_{n+1}|}{|[-q_n - q_{n+1}, 0]|} \geq C_2 C_3 C_4.$$

We observe that C_3 comes from Proposition 2 and C_4 from Lemma 4.2.

2^{nd} step. We show that $\eta_2 \geq K_2 \beta_{n-1} \alpha_n$.

By Lemma 4.1 there exists a constant $C_5 > 0$ such that:

$$\eta_2 \geq C_5 \frac{|(-q_{n+1}, -q_{n+1} + q_n)|}{|(-q_{n+1}, -q_{n+1} + q_n)|} \geq C_5 \frac{|(-q_{n+1}, -q_{n+1} + q_n)|}{|(-q_{n+1}, -q_{n+1} + q_n)|} \frac{|(-q_{n+1} + q_n, -q_{n+1} + 2q_n)|}{|[-q_{n+1} + q_n, -q_{n+1} + 2q_n]|}.$$

After $(a_{n+1} - 2)q_n$ iterates, by the properties of distortion of Cross-ratio **Cr** we have that:

$$\begin{aligned} \eta_2 &\geq C_6 \beta_{n-1} \frac{|(-q_{n-1} - 2q_n, -q_{n-1} - q_n)|}{|(\underline{0}, -q_{n-1} - q_n)|} \\ &\geq C_6 \beta_{n-1} \frac{|-q_n|}{|[-q_n, -q_{n-1} - 2q_n]|} \frac{|(-q_{n-1} - 2q_n, -q_{n-1} - q_n)|}{|(-q_n, -q_{n-1} - q_n)|}. \end{aligned}$$

Applying the properties of distortion of cross-ratio **Cr** after q_n iterates and by Lemma 4.2 and Proposition 2 we find that, for n large enough:

$$\eta_2 \geq C_7 \beta_{n-1} \frac{|(-q_{n-1} - q_n, -q_{n-1})|}{|[-q_{n-1} - q_n, -q_{n-1})|} \geq C_7 \beta_{n-1} \frac{|(-q_{n-1} - q_n, -q_{n-1})|}{|[-q_{n-1} - q_n, -q_{n-1})|}.$$

It remains to find a bound for $\frac{|(-q_{n-1} - q_n, -q_{n-1})|}{|[-q_{n-1} - q_n, -q_{n-1})|}$. First we apply Fact 2.4 and Proposition 1 to

$$\begin{aligned} - T &= [-q_{n-1} - q_{n-2} + 1, -q_{n-1} + 1], \\ - J &= (-q_{n-1} - q_{n-2} + 1, -q_{n-1} + 1), \\ - &f^{q_{n-1}-1}. \end{aligned}$$

to have two constants $C_8 > 0$ and $C_9 > 0$ such that:

$$\begin{aligned} \frac{|(-q_{n-1} - q_n, -q_{n-1})|}{|[-q_{n-1} - q_n, -q_{n-1})|} &\geq C_8 \frac{|(-q_{n-1} - q_n + 1, -q_{n-1} + 1)|}{|[-q_{n-1} - q_n + 1, -q_{n-1} + 1)|} \\ &\geq C_9 \frac{|(-q_n, 0)|}{|[-q_n, 0)|} \geq C_9 \alpha_n. \end{aligned}$$

In order to conclude the proof, it is sufficient put together the bounds found for η_1 and η_2 . \square

Propositions 4 and 5 give us the following important inequality:

Theorem 4.15. *There exists a positive constant K such that, for n large enough*

$$\beta_n \geq K \beta_{n-1}^{\left(\frac{1}{\ell} + \frac{1-\ell^{-a_{n+1}}}{\ell-1}\right)} \beta_{n-2}^{\ell^{-a_n}}.$$

To end the proof of the second part of Theorem 1.2, by Remark 4.9, Proposition 3 and Proposition 4 it is sufficient to prove that the sequence $(\beta_n)_{n \in \mathbb{N}}$ is bounded away from zero for $\ell \geq 3$. It remains to analyze the recurrence of the sequence $(\beta_n)_{n \in \mathbb{N}}$.

Analyses of the Recurrence. We define for all $n \in \mathbb{N}$ the quantity

$$\nu_n = -\ln \beta_n$$

Theorem 4.15 implies that there exists a constant $K_1 > 0$ such that:

$$\nu_n - \left(\frac{1}{\ell} + \frac{1-\ell^{-a_{n+1}}}{\ell-1}\right) \nu_{n-1} - \ell^{-a_n} \nu_{n-2} \leq K_1. \quad (4.16)$$

We prove that the sequence $(\nu_n)_{n \in \mathbb{N}}$ is bounded. In order to do this we start to consider the sequence of vectors $(v_n)_{n \in \mathbb{N}}$:

$$v_n = \begin{pmatrix} \nu_n \\ \nu_{n-1} \end{pmatrix},$$

the sequence of matrices $(A_\ell(n))_{n \in \mathbb{N}}$:

$$A_\ell(n) = \begin{pmatrix} \frac{1}{\ell} + \frac{1-\ell^{-a_{n+1}}}{\ell-1} & \ell^{-a_n} \\ 1 & 0 \end{pmatrix}$$

and the vector

$$k = \begin{pmatrix} K_1 \\ 0 \end{pmatrix}.$$

Now we can write (4.16) in the form of

$$v_n \leq A_\ell(n) A_\ell(n-1) \dots A_\ell(2) v_1 + \left(\sum_{i=2}^{n-1} A_\ell(n-1) A_\ell(n-2) \dots A_\ell(i) \right) k \quad (4.17)$$

where the inequality must be read component-wise.

To prove that the sequence $(v_n)_{n \in \mathbb{N}}$ is bounded and then prove the second part of Theorem 1.2, it is necessary to study for all $n \in \mathbb{N}$ and $2 \leq i < n$ each product,

$$\prod_{j=i}^n A_\ell(j)$$

In particular we will estimate $\|\prod_{j=i}^n A_\ell(j)\|_\infty$.

Remark 4.18. Recall that if $A = (a_{i,j})_{1 \leq i,j \leq n}$ is a matrix, then

$$\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{1 \leq j \leq n} |a_{i,j}|$$

is an operator norm. We observe in fact that

$$\|A\|_\infty = \max_{v \in \mathbb{R}^n \setminus \{0\}} \frac{\|Av\|_\infty}{\|v\|_\infty}$$

where $\|v\|_\infty = \max_{1 \leq i \leq n} |v_i|$ if $v = (v_1, \dots, v_n)$.

Moreover if $A = (a_{i,j})_{1 \leq i,j \leq n}$ and $B = (b_{i,j})_{1 \leq i,j \leq n}$ and for all $(i,j) \in \{1, \dots, n\}^2$, $a_{ij} \leq b_{ij}$, we use the shortcut notation $A \leq B$. In this case we have that $\|A\|_\infty \leq \|B\|_\infty$.

We fix now n and i such that $2 \leq i < n$, and for all $i \leq j \leq n$ we denote $b_j = \ell^{-a_{j+1}}$, hence

$$A_\ell(j) = \begin{pmatrix} \frac{1}{\ell} + \frac{1-b_j}{\ell-1} & b_{j-1} \\ 1 & 0 \end{pmatrix}$$

we observe that the sequence of reals positives numbers $(b_j)_{j \in \mathbb{N}}$ is bounded by $1/\ell$.

We fix un integer $M > 1$. Then for all $j \in \{i, n-1\}$ we can have three different cases:

1. $a_{j+1} < M$ and $a_j < M$,
2. $a_{j+1} \geq M$,
3. $a_j \geq M$.

In the first case we denote $B := A_\ell(j)$, in the second one $U_1 := A_\ell(j)$ and in the third one $U_2 := A_\ell(j)$.

We observe that, by point (1) we cannot find products of the type BU_1B or BU_2B , because the two matrices B affect the central matrix.

We fix now $\ell \geq 3$, $j \in \{i, n-1\}$ and we consider the different combinations of matrices B , U_1 and U_2 which we can have.

- We start to consider a product of matrices of type $BBB \cdots BB$.

We observe that if $a_{j+1} < M$ and $a_j < M$ then by estimations in [4]

$$B = A_\ell(j) = \begin{pmatrix} \frac{1-b_j}{\ell-1} & b_{j-1} \\ 1 & 0 \end{pmatrix}$$

Since $\ell \geq 3$, $b_{j-1} \leq \frac{1}{3}$ and $b_j < 1$, then:

$$B \leq \begin{pmatrix} \frac{1}{2} & \frac{1}{3} \\ 1 & 0 \end{pmatrix}$$

Calculating the spectral radius of $W = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} \\ 1 & 0 \end{pmatrix}$, we find that it is $\rho(W) < 1$, then there exists $0 < \lambda_1 < 1$, and $C > 0$ such that

$$\|W^n\|_\infty < C\lambda_1^n$$

In particular, $\lim_{n \rightarrow \infty} \|W^n\|_\infty = 0$.

In conclusion, if we consider a product of length s of type $BBB \cdots BBB$, then there exists $0 < \lambda_1 < 1$ and $C_1 > 0$ such that

$$\|B^s\|_\infty \leq C_1 \lambda_1^s. \quad (4.19)$$

- We study now the product of matrices of type $U_{i_1} \cdots U_{i_{s-1}} U_2$ with $i_1, \dots, i_{s-1} \in \{1, 2\}$.

We observe that if $a_j \geq M$, then there exists $\epsilon > 0$ such that

$$U_2 = A_\ell(j) \leq \begin{pmatrix} \frac{1}{\ell} + \frac{1-b_j}{\ell-1} & \epsilon \\ 1 & 0 \end{pmatrix} \leq \begin{pmatrix} \frac{1}{3} + \frac{1-b_j}{2} & \epsilon \\ 1 & 0 \end{pmatrix}. \quad (4.20)$$

We continue to work in the limit case supposing that $\epsilon = 0$ and studying the different possibilities of length two (recall always that $\ell \geq 3$ and $0 < b_j \leq \frac{1}{3}$).

– If $a_j \geq M$ and $a_{j+2} \geq M$, then

$$\begin{aligned} U_1 U_2 &\leq \begin{pmatrix} \frac{1}{\ell} + \frac{1-b_{j+1}}{\ell-1} & b_j \\ 1 & 0 \end{pmatrix} U_2 \leq \begin{pmatrix} \frac{5}{6} & b_j \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{3} + \frac{1-b_j}{2} & 0 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{25}{36} + \frac{7}{12} b_j & 0 \\ \frac{5}{6} - \frac{b_j}{2} & 0 \end{pmatrix} \leq \begin{pmatrix} \frac{32}{36} & 0 \\ \frac{5}{6} & 0 \end{pmatrix} \\ &\leq \frac{8}{9} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

In particular

$$\|U_1 U_2\|_\infty \leq \frac{8}{9}. \quad (4.21)$$

– If $a_j \geq M$ and $a_{j+1} \geq M$ then:

$$U_2 U_2 \leq \begin{pmatrix} \frac{5}{6} & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{5}{6} & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{25}{36} & 0 \\ \frac{5}{6} & 0 \end{pmatrix} = \frac{5}{6} U_2,$$

$$\|U_2 U_2\|_\infty \leq \frac{5}{6} \quad (4.22)$$

We consider now the product of s matrices of the form $U_{i_1} \cdots U_{i_{s-1}} U_2$ with $i_1, \dots, i_{s-1} \in \{1, 2\}$. We observe that we can find at most $\frac{s}{2}$ for s even ($\frac{s-1}{2}$ for s odd) couples of the type $U_1 U_2$ and $U_2 U_2$. What remains is some isolated U_2 which has norm equal to 1 (we observe that we can't have some isolated U_1 because of $U_1 U_1 = U_2 U_1$).

So, by (4.21) and (4.22) there exists $0 < \lambda_2 < 1$ such that, if s is even

$$\|U_{i_1} \cdots U_{i_{s-1}} U_2\|_\infty \leq (\lambda_2)^{\frac{s}{2}} \quad (4.23)$$

and if s is odd

$$\|U_{i_1} \cdots U_{i_{s-1}} U_2\|_\infty \leq (\lambda_2)^{\frac{s-1}{2}} \quad (4.24)$$

- We consider now the case of a product of n_1 matrices $A_\ell(j_{n_1})A_\ell(j_{n_1-1})\cdots A_\ell(j_1)$ such that $a_{j_1}, a_{j_{n_1}+1} \geq M$ and $a_{j_2}, a_{j_3}, \dots, a_{j_{n_1}} < M$ (we are considering products of the type $U_1BB\cdots BB U_2$) Under these hypotheses, in the limit case we have that

$$U_2 = A_\ell(j_1) \leq \begin{pmatrix} \frac{5}{6} & 0 \\ 1 & 0 \end{pmatrix},$$

$$U_1 = A_\ell(j_{n_1}) \leq \begin{pmatrix} \frac{5}{6} & \frac{1}{3} \\ 1 & 0 \end{pmatrix}$$

and for all $k \in \{2, n_1 - 1\}$

$$B = A_\ell(j_k) \leq \begin{pmatrix} \frac{1}{2} & \frac{1}{3} \\ 1 & 0 \end{pmatrix}$$

We assume that n_1 is even and $n_1 > 2$. We have that:

$$\begin{aligned} U_1 B \dots B U_2 &\leq \begin{pmatrix} \frac{5}{6} & \frac{1}{3} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{3} \\ 1 & 0 \end{pmatrix}^{n_1-4} \begin{pmatrix} \frac{1}{2} & \frac{1}{3} \\ 1 & 0 \end{pmatrix}^2 \begin{pmatrix} \frac{5}{6} & 0 \\ 1 & 0 \end{pmatrix} \\ &\leq \frac{5}{6} \begin{pmatrix} \frac{5}{6} & \frac{1}{3} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{3} \\ 1 & 0 \end{pmatrix}^{n_1-4} \begin{pmatrix} \frac{1}{2} & \frac{1}{3} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \\ &\leq \frac{5}{6} \begin{pmatrix} \frac{5}{6} & \frac{1}{3} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{3} \\ 1 & 0 \end{pmatrix}^{n_1-4} \begin{pmatrix} \frac{5}{6} & 0 \\ 1 & 0 \end{pmatrix} \\ &\leq \left(\frac{5}{6}\right)^{\frac{n_1-2}{2}} \begin{pmatrix} \frac{5}{6} & \frac{1}{3} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{5}{6} & 0 \\ 1 & 0 \end{pmatrix} \\ &\leq \left(\frac{5}{6}\right)^{\frac{n_1-2}{2}} \begin{pmatrix} \frac{37}{36} & 0 \\ 1 & 0 \end{pmatrix} \\ &\leq \frac{37}{36} \frac{5}{6} \left(\frac{5}{6}\right)^{\frac{n_1-4}{2}} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \\ &\leq \left(\frac{8}{9}\right)^{\frac{n_1-2}{2}} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

We assume now that n_1 is odd, $n_1 \geq 3$, then:

$$\begin{aligned} U_1 B \dots B U_2 &\leq \begin{pmatrix} \frac{5}{6} & \frac{1}{3} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{3} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{3} \\ 1 & 0 \end{pmatrix}^{n_1-3} \begin{pmatrix} \frac{5}{6} & 0 \\ 1 & 0 \end{pmatrix} \\ &\leq \left(\frac{5}{6}\right)^{\frac{n_1-3}{2}} \begin{pmatrix} \frac{5}{6} & \frac{1}{3} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{3} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{5}{6} & 0 \\ 1 & 0 \end{pmatrix} \\ &\leq \left(\frac{5}{6}\right)^{\frac{n_1-1}{2}} \begin{pmatrix} \frac{5}{6} & \frac{1}{3} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \\ &\leq \frac{7}{6} \frac{5}{6} \left(\frac{5}{6}\right)^{\frac{n_1-3}{2}} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \\ &\leq \left(\frac{35}{36}\right)^{\frac{n_1-1}{2}} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

So we can conclude that for each sequence of n_1 matrices $U_1 B \dots B U_2$, there exists $0 < \lambda_3 < 1$ such that

$$\|U_1 B B \dots B B U_2\|_\infty \leq \lambda_3^{\frac{n_1-2}{2}}. \quad (4.25)$$

After these first observations, we fix $n \in \mathbb{N}$, and $2 \leq i \leq n-1$ and we estimate the following quantity:

$$\left\| \prod_{j=i}^n A_\ell(j) \right\|_\infty.$$

Lemma 4.26. *Let $n \in \mathbb{N}$, $2 \leq i \leq n-1$ and $A := \prod_{j=i}^n A_\ell(j)$.*

Then there exists a constant $s > 0$ such that for each term $A_\ell(k) \circ \dots \circ A_\ell(k-s)$ of A of length s ,

$$\|A_\ell(k) \dots A_\ell(k-s)\|_\infty \leq \omega$$

with $\omega \in (0, 1)$.

Proof. We consider a term $A_\ell(k_r) \circ \dots \circ A_\ell(k_1)$ of A .

- If for all $k_1 < j < k_r$ $A_\ell(j) = B$, then by (4.19) there exist two constants $C > 0$ and $\lambda_1 \in (0, 1)$ such that:

$$\|A_\ell(k_r) \dots A_\ell(k_1)\|_\infty \leq C \lambda_1^r.$$

We observe that we have the same estimation pour a sequence of the type $U_1 \underbrace{B \dots B}_r U_2$.

- If $A_\ell(k_r) \circ \dots \circ A_\ell(k_1)$ is composed by a first sequence of the type $B \dots B U_2$ by sequences of the type $U_1 B \dots B U_2$ or $U_{\{1,2\}} \dots U_2$ and by a last sequence of the type $U_1 B \dots B$, then by (4.19), (4.23), (4.24) and (4.25) there exists $C > 0$ and $\lambda_2 \in (0, 1)$ such that $\| \underbrace{B B \dots B B}_{n_1} U_2 \underbrace{\dots}_{n_2} \dots \underbrace{\dots}_{n_k} U_1 \underbrace{B B \dots B B}_{n_{k+1}} \|_\infty$ is less or equal

than

$$C \lambda_2^{n_1} \lambda_2^{n_2} \dots \lambda_2^{n_k} C \lambda_2^{n_{k+1}} = C^2 \lambda_2^{n_1 + n_2 + \dots + n_k + n_{k+1}} = C^2 \lambda_2^{\tilde{r}}.$$

We denote now $\lambda = \max\{\lambda_1, \lambda_2\}$ and $s > 0$ such that $\max\{C \lambda^s, C^2 \lambda^s\} < \omega < 1$.

□

Finally, by Lemma 4.26 there exist $s > 0$ and $\omega \in (0, 1)$ such that, if $\prod_{j=i}^n A_\ell(j)$ contains k term of length s , then:

$$\left\| \prod_{j=i}^n A_\ell(j) \right\|_\infty \leq C \omega^k \quad (4.27)$$

with the constant $C > 0$ which is the upper bound for the last term of length $n-i-k s+1$.

We observe that the estimations found are valid also in the limit case, we have in fact used the hypothesis that $\epsilon = 0$ (see (4.20)). In the general case, we can use the continuity of the norm $\|\cdot\|_\infty$ as a function of ϵ in order to have the same type of estimation than in (4.27).

In conclusion, by inequality (4.17), the sequence $\nu_n = -\log \beta_n$ is bounded and by Remark 4.9 and Propositions 3 and 4, the proof of second part of Theorem 1.2 is complete.

5 On Physical Measures for Cherry Flows

Basic definitions and observations. Let ϕ be a continuous flow on a compact manifold M . A probability measure ν is *invariant* under the flow if $\nu(\phi_t(A)) = \nu(A)$, for all $t \in \mathbb{R}$ and for any measurable set $A \subset M$.

Definition 5.1. Let $t > 0$, we define the following family of probability measures $m_t(z)$ by:

$$\int_M \alpha dm_t(z) = \frac{1}{t} \int_0^t \alpha(\phi_s(z)) ds,$$

for each continue function $\alpha : M \rightarrow \mathbb{R}$.

If we take a measure μ , its basin of attraction $B(\mu) = B^\phi(\mu)$ is the set of $z \in M$ such that:

$$\lim_{t \rightarrow \infty} m_t(z) = \mu \text{ (for the weak-} \star \text{ topology)}$$

The measure μ is said physical if its basin of attraction has strictly positive Lebesgue measure.

We are interested in the study of physical measures for Cherry flows. If we consider a Cherry flow as an analytic flow on the two-dimensional torus without closed orbits and with two singularities, a sink and a saddle (like in the originally construction of Cherry in [2]), the situation is clear: there is just one physical measure which is the Dirac delta at the sink. The scenario become more interesting if we consider a \mathcal{C}^∞ flow ϕ on the torus \mathbb{T}^2 , without closed orbits and with two singularities, a saddle point p_s and a repelling point p_r , both hyperbolic. It is still called Cherry flow.

Let g be the first return map of the flow ϕ on the Poincaré section⁴. In this case g is \mathcal{C}^∞ everywhere except at one point which belongs to the stable manifold of the saddle point and which we will assume to be zero (we identify \mathbb{S}^1 with $[-\frac{1}{2}, \frac{1}{2}]_{-\frac{1}{2} \sim \frac{1}{2}}$). We denote by a and b respectively the left-sided limit and the right-sided limit of the discontinuity point and by $U = (a, b)$

Definition 5.2. Let z be a point of the Poincaré section. The first return time $\tau(z)$ for ϕ to \mathbb{S}^1 is the number of iteration of g needed by z to come back to \mathbb{S}^1 for the first time.

Fact 5.3. The first return time $\tau(z)$ for ϕ to \mathbb{S}^1 has a logarithmic singularity in 0. This means that for all $\epsilon > 0$, there exists a constant $C > 0$ such that, for all z in the interval $(-\epsilon, \epsilon)$, we have that $\frac{1}{C} \leq \frac{\tau(z)}{-\log|z|} \leq C$. In other word $\tau(z)$ is of order of $-\log|z|$.

We consider now the flow φ obtained reversing the direction of ϕ . The repelling point of ϕ becomes an attractive point for φ which is then a Cherry flow like in Cherry's example ([2]). In this case, the first return map f of φ to the Poincaré section belongs to the class \mathcal{L} with flat interval $U = (a, b)$ (for more details see [1], [10], [11]). Moreover if $\lambda_1 > 0 > \lambda_2$ are the eigenvalues of the saddle point p_s of ϕ , then the critical exponent of f will be $\ell = \frac{\lambda_1}{-\lambda_2}$.

Since the considered flow has not closed orbits, then f has an irrational rotation number ρ . Moreover, f preserves the order and it has not periodic points, so by Poincaré's

⁴For the construction of the first return map and of the Poincaré section the reader can refer to [10] or [1].

Theorem, there exists a continuous preserving order function $h : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ such that $h \circ f = R_\rho \circ h$, where R_ρ is the rotation by ρ .

So, the first return function f has only one invariant probability measure μ supported on the minimal set (which is an invariant measure also for g), given by $\mu = h^*(\text{Leb})$. By Proposition 2 in [12] this measure μ can be extended to a probability invariant measure ν on the torus supported on the quasi-minimal set depending on the fact that $\int_{\mathbb{S}^1} \tau d\mu$ is convergent or not. In [13] the authors prove that this is the case when the divergence at the saddle point is positive.

Assumptions. Let 0 be the discontinuity point of g . Following the notation of the previous sections, we denote by:

- $\underline{i} = f^i(0)$,
- $\underline{i}_R = R_\rho^i(0)$.

5.1 Proof of Theorem 1.3

We recall that:

$$\alpha_n = \frac{|(-q_n, \underline{0})|}{|[-q_n, \underline{0})|}.$$

We prove the following proposition:

Proposition 6. *Let f be the first return function to φ ⁵. If f has rotation number of bounded type and critical exponent $\ell \in (1, 2]$, then there exist two constants $K > 0$ and $C < 1$ such that for n big enough, $\frac{-\log \alpha_n}{q_{n+1}} \leq KC^n$. This constant C doesn't depend on the critical exponent ℓ of f .*

Proof. Before to begin the proof it is necessary to recall that:

1. by the recursive formula (1.1) $q_0 = 1$, $q_1 = a_1$ and $q_{n+1} = a_{n+1}q_n + q_{n-1}$,
2. by Proposition 6 in [4], for n big enough $\alpha_n \geq \alpha_{n-1}^{\frac{1-\ell^{-a_{n+1}}}{\ell-1}} \alpha_{n-2}^{\ell^{-a_n}}$.

To lighten the notation, we introduce a new sequence $(\theta_n)_{n \in \mathbb{N}}$ defined for all n , by $\theta_n := -\log \alpha_n$.

We fix $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, the point (2) is verified.

We shall prove the proposition by induction on n . We take $C = C(\ell) = \sup_i \left(\frac{1-\ell^{-a_i}}{(\ell-1)a_i} \right)^{\frac{1}{n_0}}$ and $K \geq \max\{\theta_{n_0-2}, \theta_{n_0-1}\}$. We observe that, for all $1 < \ell \leq 2$, we have $C < 1$; if we consider C as a function of ℓ , then, in the interval $(1, 2]$, $C(\ell)$ is continuous, non-decreasing and moreover $\lim_{\ell \rightarrow 1} C(\ell) = 1$ and $C(2) < 1$.

We observe that, for any natural number $i \geq 1$

$$\frac{1 - \ell^{-a_i}}{(\ell - 1)a_i} \leq C \tag{5.4}$$

and

$$\ell^{-a_i} \leq \frac{1 - \ell^{-a_i}}{(\ell - 1)a_i} \leq C^{n_0}. \tag{5.5}$$

We begin now the proof by induction.

⁵Recall that $f \in \mathcal{L}$.

- Let n_0 as above. By point (2) and by (5.5), we have that:

$$\begin{aligned}\theta_{n_0} &\leq \left(\frac{1 - \ell^{-a_{n_0+1}}}{\ell - 1} \right) \theta_{n_0-1} + \ell^{-a_{n_0}} \theta_{n_0-2} \\ &\leq C^{m_0} K a_{n_0+1} + C^{m_0} K \\ &\leq K C^{m_0} (a_{n_0+1} + 1).\end{aligned}$$

By point (1) :

$$\theta_{n_0} \leq K C^{m_0} q_{n_0+1}. \quad (5.6)$$

- We prove now the assertion for $n_0 + 1$. By point (2) and by (5.6) and (5.5):

$$\begin{aligned}\theta_{n_0+1} &\leq \left(\frac{1 - \ell^{-a_{n_0+2}}}{\ell - 1} \right) \theta_{n_0} + \ell^{-a_{n_0+1}} \theta_{n_0-1} \\ &\leq K C^{m_0} \left(\frac{1 - \ell^{-a_{n_0+2}}}{(\ell - 1) a_{n_0+2}} \right) a_{n_0+2} q_{n_0+1} + K C^{m_0}.\end{aligned}$$

And by point (1) and (5.4) :

$$\begin{aligned}\theta_{n_0+1} &\leq K C^{m_0+1} \left(a_{n_0+2} q_{n_0+1} + \frac{C^{m_0}}{C} \right) \\ &\leq K C^{m_0+1} q_{n_0+2}.\end{aligned}$$

- We assume now that the assertion is true for $n - 2$ and for $n - 1$ and we prove it for n .

By point (2) and the inductive hypothesis we have that:

$$\begin{aligned}\theta_n &\leq \frac{1 - \ell^{-a_{n+1}}}{\ell - 1} \theta_{n-1} + \ell^{-a_n} \theta_{n-2} \\ &\leq K \left(\frac{(1 - \ell^{-a_{n+1}})}{(\ell - 1) a_{n+1}} C^{n-1} a_{n+1} q_n + C^{n-2} \ell^{-a_n} q_{n-1} \right).\end{aligned}$$

Finally, by (5.4), (5.5) and by point (1)

$$\begin{aligned}\theta_n &\leq K C^n \left(a_{n+1} q_n + \frac{\ell^{-a_n}}{C^2} q_{n-1} \right) \\ &\leq K C^n q_{n+1}.\end{aligned}$$

So, the assertion of the lemma is true for all $n \in \mathbb{N}$ big enough. □

We recall the following theorem proved in [4] :

Theorem 5.7. *Let f be the first return function for φ with critical exponent $\ell > 1$ ($\lambda_1 > -\lambda_2$), then $\cup_{i=0}^{\infty} f^{-i}(U)$ ⁶ has full Lebesgue measure on \mathbb{S}^1 .*

The proof of Theorem 1.3 uses the mains ideas of the proof of Theorem 3 in [13].

⁶Observe that $\cup_{i=0}^{\infty} f^{-i}(U)$ is the wandering set of φ .

Proof of Theorem 1.3. By Theorem 2 in [13] we know that the flow ϕ has a invariant probability measure ν supported on the quasi-minimal set which correspond to the extension of the f -invariant measure μ (defined by $\mu = h^*(\text{Leb})$). It remains to prove that ν is a physical measure for ϕ and that its basin of attraction has full Lebesgue measure.

By Theorem 5.7, it is sufficient to prove that the points of the wandering set of φ are in the basin of attraction of ν . Since all points of the wandering set pass through the flat interval of f , then we have just to prove that any point of U is in the basin of attraction of ν .

Let $z \in U$, $\underline{n}_g = g^{n-1}(z)$ and $t_n = \tau(\underline{n}_g)$. For all $t > 0$ there exists $N \in \mathbb{N}$ such that $t = t_1 + t_2 + \dots + t_N + \tilde{t}$ where $0 < \tilde{t} \leq t_{N+1}$. Moreover, let $n \in \mathbb{N}$ such that $q_n \leq N < q_{n+1}$. Since τ is uniformly minored we have that:

$$t \geq CN \quad (5.8)$$

with C a positive constant.

Let m_t be the probability measure on the orbit of flow of length $t > 0$ which starts in z (see Definition 5.1).

Since the only invariant probability measures are δ_s , δ_r and ν and since b is repelling, then the limits for m_t will be of the form

$$r\delta_s + (1-r)\nu, \quad (5.9)$$

for some $r \in [0, 1]$. In order to prove that z is in the basin of attraction of ν , which means that $\lim_{t \rightarrow \infty} m_t = \nu$, it is necessary to prove that $r = 0$.

We fix $0 < n_0 < n$ and we prove that the trajectory of z under ϕ spends the most of the time outside of

$$A_{n_0} = \{\phi_s(w) : w \in (\underline{q}_{n_0}, \underline{q}_{n_0} + 1), 0 \leq s \leq \tau(w)\}.$$

We prove that, choosing correctly n_0 , the time $t_{A_{n_0}}$ which the trajectory $\phi_s(z)$, $0 \leq s \leq t$ spends in A_{n_0} can be done arbitrarily small in comparison to t , for all t big enough.

In order to do this we divide A_{n_0} and we start to estimate the time t_{B_l} spent by the trajectory $\phi_s(z)$, $0 \leq s \leq t$ in

$$B_l = \{\phi_s(w) : w \in (\underline{q}_l, \underline{q}_{l+2}), 0 \leq s \leq \tau(w)\}.$$

We observe that, since f is the first return function of the flow obtained reversing the direction of ϕ , if h is the semi-conjugation between f and the rotation R_ρ , then

$$h(\underline{n}_g) = h(g^{n-1}(z)) = h(f^{-n+1}(z)) = h(f^{-n}(0)) = \underline{n}_R.$$

Then for all $l \in \mathbb{N}$ we have $\underline{q}_g \in (\underline{q}_{l-1}, \underline{q}_{l+1})$ and $\underline{q}_l \in (\underline{q}_{l-1_g}, \underline{q}_{l+1_g})$. So we can say that the number of the points \underline{i}_q , $1 \leq i \leq N$ in $(\underline{q}_l, \underline{q}_{l+2})$ is equal to the number of the points \underline{i}_R , $1 \leq i \leq N$ in $(\underline{q}_R, \underline{q}_{l+2_R})$.

So we estimate the number N_l of the points \underline{i}_R , $1 \leq i \leq N$ which are in $(\underline{q}_R, \underline{q})$. Since $|(\underline{q}_R, 0)|$ is of the order of $\frac{1}{q_{l+1}}$ and since the rotation is a bijection preserving the distance, then we can divide the circle in exactly q_{l+1} disjoint images of $(\underline{q}_R, 0)$ and any image has N_l points \underline{i}_R , $1 \leq i \leq N$.

In conclusion

$$q_{l+1}N_l \leq N$$

and the number of the points $\underline{-i}_R$, $1 \leq i \leq N$ which are in $(\underline{q}_l, \underline{q}_{l+2})$ is lesser than or equal to $\frac{N}{q_{l+1}}$

By Fact 5.3, Equation (5.8) and Proposition 6⁷ we have that:

$$\begin{aligned} \frac{t_{A_{n_0}}}{t} &= \frac{1}{t} \sum_{l=n_0}^{n-1} t_{B_l} \leq \frac{C_3 N}{t} \sum_{l=n_0}^{n-1} \frac{-\log |(\underline{q}_{l+2}, \underline{0})|}{q_{l+1}} \\ &\leq \frac{C_3}{C} \sum_{l=n_0}^{n-1} \frac{-\log |(\underline{q}_{l+2}, \underline{0})|}{q_{l+1}} \\ &\leq \frac{C_3}{CC_4} \sum_{l=n_0}^{n-1} (C_5)^{l+2}. \end{aligned}$$

Finally, since $\sum_{l=n_0}^{\infty} (C_6)^l$ is convergent, taking n_0 big enough, we can make $\frac{t_{A_{n_0}}}{t}$ as small as we want.

We observe that we have the same result if in place of A_{n_0} we consider A_{n_0-c} with $c > 0$ and $A_{n_0} \Subset A_{n_0-c}$.

It remains to prove that if $\lim_{n_0 \rightarrow \infty} \frac{t_{A_{n_0}}}{t} = 0$ then $r = 0$.

We suppose by contradiction that $r > 0$ and we recall that there exists a sequence strictly non-decreasing $(t_n)_{n \in \mathbb{N}}$, $t_n \rightarrow +\infty$, when $n \rightarrow +\infty$ of positive reals, such that: $\lim_{t_n \rightarrow \infty} m_{t_n}(z) = r\delta_s + (1-r)\nu$ (see (5.9)).

So there exists $T > 0$ such that for all $n \in \mathbb{N}$ for which $t_n > T$ and for any $\alpha : \mathbb{T}^2 \rightarrow \mathbb{R}$ continue

$$\left| \int_{\mathbb{T}^2} \alpha dm_{t_n}(z) - \int_{\mathbb{T}^2} \alpha d(r\delta_s + (1-r)\nu) \right| < \epsilon. \quad (5.10)$$

Let now $c > 0$ be such that $A_{n_0} \Subset A_{n_0-c}$. Let α be a bump function with compact support such that, for all $x \in A_{n_0}$, $\alpha(x) = 1$ and for all $x \in (A_{n_0-c})^c$, $\alpha(x) = 0$. We observe that

$$\frac{t_{A_{n_0}}}{t_n} \leq \int_{\mathbb{T}^2} \alpha dm_{t_n}(z) \leq \frac{t_{A_{n_0-c}}}{t_n}$$

and we recall that by hypothesis $\lim_{n_0 \rightarrow \infty} \frac{t_{A_{n_0}}}{t_n} = \lim_{n_0 \rightarrow \infty} \frac{t_{A_{n_0-c}}}{t_n} = 0$.

Then, for n big enough, we deduce by (5.10),

$$r - \epsilon < r + (1-r)\nu(A_{n_0}) - \epsilon < \epsilon \quad (5.11)$$

which is in contradiction with the hypothesis that $r > 0$.

So $r = 0$ and (for the weak- \star topology) $\lim_{t \rightarrow \infty} m_t(z) = \nu$. By Definition 5.1, z is in the basin of attraction of ϕ . \square

5.2 Proof of Theorem 1.4

The idea of the proof of this theorem is quite the same of the proof of Theorem 1.3. The main technical tool is the existence of the bounded geometry for the first return function f with critical exponent $\ell \geq 3$, without any assumption on the rotation number (the second part of Theorem 1.2).

⁷We need to a supplementary hypotheses on the eigenvalues $\lambda_1 > 0 > \lambda_2$ of the saddle point. We assume en fact that $\lambda_1 \leq -2\lambda_2$. The case $\lambda_1 > -2\lambda_2$ is proved in [13] (see Theorem 3).

Proof of Theorem 1.4. The condition $\lambda_1 \geq -3\lambda_2$ is equivalent to the fact that the critical exponent of the first return function f is greater than or equal to 3. So, we are in the hypothesis of the second part of Theorem 1.2 and we can assume that exist $n_0 \in \mathbb{N}$ and a constant $\alpha \in (0, 1)$ such that $\frac{|(\underline{0}, \underline{q}_n)|}{|(\underline{0}, \underline{q}_{n-2})|} > \alpha^2$ for $n \geq n_0 > 0$. Then, by induction

$$|(\underline{0}, \underline{q}_n)| > C\alpha^n \quad (5.12)$$

for some $C > 0$.

By Theorem 2 in [13], there exists an invariant probability measure ν supported on the quasi-minimal set. We prove that the basin of attraction of ν , has full Lebesgue measure; so ν is a physical measure for ϕ .

Like in the proof of Theorem 1.3 we prove that any point of U is in the basin of attraction of ν .

Let $z \in U$, $\underline{n}_g = g^{n-1}(z)$ and $t_n = \tau(\underline{n}_g)$. For all $t > 0$ there exists $N \in \mathbb{N}$ such that $t = t_1 + t_2 + \dots + t_N + \tilde{t}$ where $0 < \tilde{t} \leq t_{N+1}$ and there exists $n \in \mathbb{N}$ such that $q_n \leq N < q_{n+1}$. Since τ is uniformly minored, we have that

$$t \geq C_1 N \quad (5.13)$$

with $C_1 > 0$ a positive constant.

Let m_t be the probability measure on the orbit of ϕ of length $t > 0$ which start at z (see Definition 5.1). The possible limits for m_t must have the form $r\delta_s + (1-r)\nu$, for some $r \in [0, 1]$. We have to prove that r is zero (for the details see the proof of Theorem 1.3).

We fix $0 < n_0 < n$ and we prove that the orbit of z under ϕ spends most of time outside of

$$A_{n_0} = \{\phi_s(w) : w \in (\underline{q}_{n_0}, \underline{q}_{n_0+1}), 0 \leq s \leq \tau(w)\}.$$

The time $t_{A_{n_0}}$ spent in A_{n_0} will be calculated as the sum of the times t_{B_l} spent in small pieces of A_{n_0} of the form

$$B_l = \{\phi_s(w) : w \in (\underline{q}_l, \underline{q}_{l+2}), 0 \leq s \leq \tau(w)\}.$$

For these reasons, it is necessary to estimate the number of the points \underline{i}_g , $1 \leq i \leq N$ in $(\underline{q}_l, \underline{q}_{l+2})$ which, like in Theorem 1.3, coincides with the number of the points $\underline{-i}_R$, $1 \leq i \leq N$ in $(\underline{q}_{l_R}, \underline{q}_{l+2_R})$ ⁸ which is less than or equal to $\frac{N}{q_{l+1}}$.

Finally by (5.12), (5.13), Fact 5.3 and the fact that $q_l \geq \frac{\beta^l}{C_6}$ for $\beta = \frac{1+\sqrt{5}}{2}$ we have that:

$$\begin{aligned} \frac{t_{A_{n_0}}}{t} &= \frac{1}{t} \sum_{l=n_0}^{n-1} t_{B_l} \leq \frac{C_5 N}{t} \sum_{l=n_0}^{n-1} \frac{-\log |(\underline{q}_{l+2}, \underline{0})|}{q_{l+1}} \\ &\leq \frac{C_5}{C_1} \sum_{l=n_0}^{n-1} \frac{-\log |(\underline{q}_{l+2}, \underline{0})|}{q_{l+1}} \\ &\leq \frac{C_5 C \alpha}{C_1} \sum_{l=n_0}^{n-1} \frac{l}{q_{l+1}} \\ &\leq \frac{C_5 C \alpha C_6}{C_1} \sum_{l=n_0}^{n-1} \frac{l}{\beta^{l+1}}. \end{aligned}$$

⁸ f is the first return function of φ obtained reversing the direction of ϕ .

In conclusion, taking n big enough, we can make $\frac{t_{A_n}}{t}$ as small as we want: then, like in the proof of Theorem 1.3, $r = 0$ and z is in the basin of attraction of ν . \square

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